# Introducing Finite Constraint Systems

DRAFT 0.8.4; Maurice Gittens <maurice at gittens dot nl>

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**Abstract**

The object of study in this article are finite constraint systems, a synthesis of ordered finite sets of elements and finite fields. The first section introduces and defines finite constraint systems and their representation. Next we will show that finite constraint systems are introduce a new class finite fields. The partitioning of finite constraint systems is the following topic presented. The next section introduces *matching* and *remainder* operations for finite sets. Next the satisfiability problem is expressed using finite constraint systems and a generalized version of the satisfiability problem will be presented.

# Contents

# 1 The Core Building Block: finite fields as finite functions f:$\mathbb{Z} \to \mathbb{Z}$

## 1.1 Definition

Let $F$ be the set of all finite fields. Each $f \in F$ defines functions f:$\mathbb{Z} \to \mathbb{Z}$ called *field functions* that map integers $x$ to integers $f(x)$ in accordance with the following recipe:

$$\begin{array}{|cccc|c|}\hline l_{0,0} & l_{0,1} & \dots & l_{0,C-1} & h_0 \\ l_{1,0} & l_{1,1} & \dots & l_{1,C-1} & h_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{R-1,0} & l_{R-1,1} & \dots & l_{R-1,C-1} & h_{R-1} \\ \hline v_0 & v_1 & \dots & v_{C-1} & (x, f(x)) \\ \hline \end{array}$$

- the field matrix $\mathbb{Z}_f^{R-1\times C-1}$ is a matrix $m$ over a finite field $f$ with $R,C$ respectively representing the number of rows and the columns in the matrix. A matrix element $l_{i,j} \in f$ will be called a literal when $l_{i,j} \neq 0$[1].
- The number of elements in the field $f$, will be called the *field radix* and is denoted |f|, $r_r = |f|^R$ is called the *row radix* and $r_c = |f|^C$ is called the *column radix* of $m$
- $v_x = \Sigma_{i=0}^{R-1} l_{i,x} |f|^i$ for each column $x$ of m
- $h_x = \Sigma_{i=0}^{C-1} l_{x,i} |f|^i$ for each row $x$ of
- $x = \sum_{i=0}^{C-1} v_i r_r^j$
- $f(x) = \sum_{i=0}^{V-1} h_i r_c^j$

## 1.2 Finite Constraint Systems

### 1.2.1 Definition

In section 1.1 all finite fields $f$ were recognized as functions $f : \mathbb{Z} \to \mathbb{Z}$. A finite constraint system is defined as a specific mapping from an $a \in \mathbb{Z}$ to it's image $f(a)$.

### 1.2.2 Example Finite Constraint Systems for the field $\mathbb{Z}/\mathbb{Z}_3$

This section presents a few examples of mapping of integers to their image using a finite constraint function for $\mathbb{Z}/\mathbb{Z}_3$. To determine the image of the integer 4, we recognize that the field radix of $\mathbb{Z}/\mathbb{Z}_3$ is equal the order of this field which is 3. For the following examples the *row radix* and the *column radix* are $27 = 3^3$ and $81 = 3^4$ respectively.

$$\mathbb{Z}/\mathbb{Z}_3(4) = \begin{array}{|cccc|c|}\hline 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 0 & (4, 244) \\ \hline \end{array}$$

$\mathbb{Z}/\mathbb{Z}_3(4) = 244$ because $4 \cdot 27^0 = 4$ and $1 \cdot 81^0 + 3 \cdot 81^1 = 244$ .

Similarly $\mathbb{Z}/\mathbb{Z}_3(310018) = 300208$ because $4 \cdot 27^0 + 7 \cdot 27^1 + 20 \cdot 27^2 + 15 \cdot 27^3 = 310018$ and $22 \cdot 81^0 + 61 \cdot 81^1 + 45 \cdot 81^2 = 300208$

$$\mathbb{Z}/\mathbb{Z}_3(310018) = \begin{array}{|cccc|c|}\hline 1 & 1 & 2 & 0 & 22 \\ 1 & 2 & 0 & 2 & 61 \\ 0 & 0 & 2 & 1 & 45 \\ \hline 4 & 7 & 20 & 15 & (310018, 300208) \\ \hline \end{array}$$

Similarly $\mathbb{Z}/\mathbb{Z}_3(14607) = 118198$ because $4 \cdot 27^0 + 20 \cdot 27^2 = 14607$ and $19 \cdot 81^0 + 1 \cdot 81^1 + 18 \cdot 81^2 = 118198$

[1] 0 is the additive identity of f

$$\mathbb{Z}/\mathbb{Z}_3(14607) = \begin{array}{|cccc|c|} \hline 1 & 0 & 2 & 0 & 19 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 18 \\ \hline 4 & 0 & 20 & 0 & (14607, 118198) \\ \hline \end{array}$$

## 1.3 More consise representation of Finite Constraint Systems

In this paper, Finite Constraint Systems will sometime be represented for consisely as in the following

$$\begin{array}{|cccc|} \hline l_{0,0} & l_{0,1} & \cdots & l_{0,C-1} \\ l_{1,0} & l_{1,1} & \cdots & l_{1,C-1} \\ \vdots & \vdots & \vdots & \vdots \\ l_{R-1,0} & l_{R-1,1} & \ldots & l_{R-1,C-1} \\ \hline \end{array}$$

The information omitted is redundate and can readily be inferred from the information available.

## 2 Finite Constraint Systems as a class of Finite Fields

This section will show that finite constraint systems are class of finite fields.

Let $f$ be a finite *field function* f:$\mathbb{Z} \to \mathbb{Z}$ with corresponding *RxC* field matrix $m$ and binary additive and multiplicative operations on finite constraint systems $s_1$ and $s_2$ are collectively denoted by $\star$.

$$s_1 \star s_2 = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,|I|-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,|I|-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{v-1,0} & a_{v-1,1} & \cdots & a_{v-1,|I|-1} \end{bmatrix} \star \begin{bmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,|I|-1} \\ b_{1,0} & b_{1,1} & \cdots & b_{1,|I|-1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{v-1,0} & b_{v-1,1} & \cdots & b_{v-1,|I|-1} \end{bmatrix} = \begin{bmatrix} a_{0,0} \star b_{0,0} & a_{0,1} \star b_{0,1} & \cdots & a_{0,|I|-1} \star b_{0,|I|-1} \\ a_{1,0} \star b_{1,0} & a_{1,1} \star b_{1,1} & \cdots & a_{1,|I|-1} \star b_{1,|I|-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{v-1,0} \star b_{v-1,0} & a_{v-1,1} \star b_{v-1,1} & \cdots & a_{v-1,|I|-1} \star b_{v-,|I|-1} \end{bmatrix}$$

To recognize these finite constraint systems as as finite fields we must recognize that

- $s_1 \star s_2$ is closed under $\star$
- the additive identity is the $R \times C$ matrix $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ where 0 represents the additive identity of $f$
- the multiplicative identity is the $R \times C$ matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ where 1 represents the multiplicative identity of $f$ of
- additive inverse of $s_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{v,1} & a_{v,2} & \cdots & a_{v,k} \end{bmatrix}$ is defined as $\begin{bmatrix} -a_{1,1} & -a_{1,2} & \cdots & -a_{1,k} \\ -a_{2,1} & -a_{2,2} & \cdots & -a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{v,1} & -a_{v,2} & \cdots & -a_{v,k} \end{bmatrix}$ where $-a_{i,i}$ is the additive inverse of $a_{i,i}$ in $f$
- multiplicative inverse of $s_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{v,1} & a_{v,2} & \cdots & a_{v,k} \end{bmatrix}$ is defined as $\begin{bmatrix} a_{1,1}^{-1} & a_{1,2}^{-1} & \cdots & a_{1,k}^{-1} \\ a_{2,1}^{-1} & a_{2,2}^{-1} & \cdots & a_{2,k}^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{v,1}^{-1} & a_{v,2}^{-1} & \cdots & a_{v,k}^{-1} \end{bmatrix}$ where $a_{i,i}^{-1}$ is the multiplicative inverse of $a_{i,i}$ in $f$
- it is also easy to recognize that operations $\star$ are both *associative* and *commutative* as these properties are obtained from $f$

The above proves that finite constraint systems for a class of finite fields.

# 3 Integer multiplication and exponentiation for finite constraint systems

## 3.1 Multiplication as repeated addition

Let $m$ be a finite constraint system, $-m$ represent the matrix $m$ with all elements replaced by their additive inverse and $m_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ be the additive identity matrix for $m$. Multiplication by integers $i$ denoted $i \times m$ is defined by distinguishing the following cases:

- $0 * m = m_0$:
- $i * m = \sum_{i=1}^{i} m$ when $i > 0$
- $i * m = \sum_{i=1}^{i} -m$ when $i < 0$

## 3.2 Exponentiation as repeated multiplication

Let $m$ be a finite constraint matrix, $m^{-1}$ represent the matrix $m$ with all elements replaced by their multiplicative inverse and $m_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ be the multiplicative identity matrix for $m$. Exponentiation by integers $i$ denoted $m^i$ is defined by distinguishing the following cases:

- $m^0 = m_1$
- $m^i = \prod_{i=1}^{i} m$ when $i > 0$
- $m = \prod_{i=1}^{i} m^{-1}$ when $i < 0$

# 4 Partitioning finite constraint systems

## 4.1 Example of column partitioning

Instances of finite matrices $m$ can be partitioned into disjunct subsets$m_0, m_1, ..., m_{2^{|k|-1}}$.

For example, consider the $\mathbb{Z}/\mathbb{Z}_3$ finite file matrix m presented below together with the partitioning key $k = \{0,4\}$.

$m = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$. Partitioning of $m$ using key $k$ denoted

$m \lhd k$ yields the the partitions $m_0$, $m_1$, $m_2$, $m_3$ defined as

$m_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ corresponds with all columns $x$ of $m$ for which $m[0,x] = 0$ and $m[4,x] = 0$. .

$m_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ corresponds with all columns $x$ of $m$ for which $m[0,x] \neq 0$ and $m[4,x] = 0$.

$m_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$ corresponds with all columns $x$ of $m$ for which $m[0,x] = 0$ and $m[4,x] \neq 0$.

$m_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ corresponds with all columns $x$ of $m$ for which $m[0,x] \neq 0$ and $m[4,x] \neq 0$.

## 4.2 Column partitioning of finite constraint systems with keys of cardinality 2

Consider a finite constraint system with matrix representation $m = \begin{array}{|cccc|c|} \hline l_{0,0} & l_{0,1} & \dots & l_{0,|I|-1} & i^e_0 \\ l_{1,0} & l_{1,1} & \dots & l_{1,|I|-1} & i^e_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{|v|-1,0} & l_{|v|-1,1} & \cdots & l_{|v|-1,|I|-1} & i^e_{|v|-1} \\ \hline i_0 & i_1 & \dots & i_{|I|-1} & (x, f(x)) \\ \hline \end{array}$ over field $F$ and

partitioning key $k = \{x,y\}, x \neq y$. The key $k$ partitions $I$ into $2^{|k|}$ possibly empty set of integers $I = i_0 \cup i_1 \cup ... \cup i_{2^k-1}$ defined as

- $i_3$ denotes the subset of columns $m$ having literals of both the row $x$ and $y$.
- $i_2$ denotes the subset of columns of $m$ that includes literals in row $x$ and necessarily including no literals in row $y$
- $i_1$ denotes the subset of columns of $m$ that with no literals of in row $x$ and necessarily including literals in row $y$.
- $i_0$ denotes the subset of columns of $m$ that included no literals in row $x$ or $y$.

### 4.3 Column partitioning keys of cardinality $k > 2$

Row partitioning with keys of cardinality greater than 2 is possible, However this generalization is not pursued further in this document.

### 4.4 Row partitioning

In the previous section column partitioning was introduced. Row partition is analogous to column partition except in the latter cases the rows of the matrix are partitioned instead of columns.

Consider a finite constraint system with matrix representation $m = \begin{array}{|cccc|c|} l_{0,0} & l_{0,1} & \cdots & l_{0,|I|-1} & i^e_0 \\ l_{1,0} & l_{1,1} & \cdots & l_{1,|I|-1} & i^e_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{|v|-1,0} & l_{|v|-1,1} & \cdots & l_{|v|-1,|I|-1} & i^e_{|v|-1} \\ \hline i_0 & i_1 & \cdots & i_{|I|-1} & (x, f(x)) \\ \hline \end{array}$ over field $F$ and partitioning key $k = \{x, y\}, x \neq y$. The key $k$ partitions $I$ into $2^{|k|}$ possibly empty set of integers $I = i_0 \cup i_1 \cup \ldots \cup i_{2^k - 1}$ defined as

- $i_3$ denotes the subset of rows $m$ having literals of both the row $x$ and $y$.
- $i_2$ denotes the subset of rows of $m$ that includes literals in row $x$ and necessarily including no literals in column $y$
- $i_1$ denotes the subset of rows of $m$ that with no literals of in row $x$ and necessarily including literals in column $y$.
- $i_0$ denotes the subset of rows of $m$ that included no literals in column $x$ or $y$.

# 5 Division and Remainder operators for Finite Constraint Systems with integers

This section is about defining division / and remainder % operators for finite constraint systems over integer sets $I$ and integers $i$ over common finite field $F$. These operators will satisfy the identity $I = I/i \cup I\,\%\,i$.

## 5.1 The division operator

### 5.1.1 A few examples

Consider

- the set of integers $I = \{166, 21, 1, 231, 101, 27, 168, 23, 153, 28\}$
- a matrix representation $m = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$ of $I$ over $\mathbb{Z}/\mathbb{Z}_3$
- the integer value 1 over $F$ represented as a single column matrix $i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The operation of filtering $I$ by $i$ denoted $I/i$ is defined as $I/i = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Similarly $I/\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $I/\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and a final example $I/\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

### 5.1.2 The general case of division with integers

The previous section presented examples of dividing finite constraint systems by integers. For the more general case we consider finite constraint systems $I, R, D$ over common finite field $F$ where the matrix representation of $D$ has only one column . The result of $I/D = R$ is defined as follows

$$\boxed{\begin{matrix} l_{0,0} & l_{0,1} & \cdots & l_{0,|I|-1} \\ l_{1,0} & l_{1,1} & \cdots & l_{1,|I|-1} \\ \vdots & \vdots & \vdots & \vdots \\ l_{|v|-1,0} & l_{|v|-1,1} & \cdots & l_{|v|-1,|I|-1} \end{matrix}} / \boxed{\begin{matrix} d_0 \\ d_1 \\ \vdots \\ d_{|v|-1} \end{matrix}} = \boxed{\begin{matrix} r_{0,0} & r_{0,1} & \cdots & r_{0,|I|-1} \\ r_{1,0} & r_{1,1} & \cdots & r_{1,|I|-1} \\ \vdots & \vdots & \vdots & \vdots \\ r_{|v|-1,0} & r_{|v|-1,1} & \cdots & r_{|v|-1,|I|-1} \end{matrix}}$$

where

- For all columns $x$

- for all rows $y$
  - $r_{y,x} = i_{y,x}$ when $d_y = 0$ or $l_{y,x} = d_y$
  - $r_{y,x} = 0$ otherwise

## 5.2 The remainder operation for finite constraint systems

### 5.2.1 The general case

Where $I$ is a finite constraint systems and $i$ is an integer, the remainder operation denoted $I\%i$, is defined as $I\%i = I - (I/i)$,where $-./$ respectively represent the set difference operator and the division operator for finite constraint systems.

### 5.2.2 A few examples

Consider the set of integers $I$ over $\mathbb{Z}/\mathbb{Z}_3$ with matrix representation $m = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$ and an integer value $i$ represented as a single column matrix $i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The operation of dividing $I$ using $i$ denoted $I\,\%\,i$ gives $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$.

Similarly $I\,\% \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$, $I\,\% \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$ and as a final example

$$I\,\% \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

# 6 The Satisfiability Problem as a finite constraint systems over $\mathbb{Z}/\mathbb{Z}_3$

## 6.1 An example

Consider the example Boolean formula $F = (x+y)\cdot(x+\tilde{y})\cdot(\tilde{x}+\tilde{z})\cdot(\widetilde{y+z})\cdot$ It is straight forward to represent $F$ as a set of integers over $\mathbb{Z}/\mathbb{Z}_3$.

$$F = \begin{array}{|cccc|} \hline 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ \hline \end{array}$$

## 6.2 The general case

Let $F = c_0 \cdot c_1 \cdot \ldots \cdot c_{k-1}$ over variable set $v$ be a well formed formula in *conjunctive normal form*.

$F$ can equivalently be represented as a matrix

$$m = \begin{array}{|cccc|} \hline l_{0,0} & l_{0,1} & \cdots & l_{0,k-1} \\ l_{1,0} & l_{1,1} & \cdots & l_{1,k-1} \\ \vdots & \vdots & \vdots & \vdots \\ l_{|v|-1,0} & l_{|v|-1,1} & \cdots & l_{|v|-1,k-1} \\ \hline \end{array}$$

and an arbitrary ordering key $k$. In particular

- $i_x \neq 0$ for all columns $x$ of $m$
- $l_{i,j} \in \{1,0,2\}$ such that $l_{i,j} = \begin{cases} 1 & l_{i,j} \in c_j \\ 0 & l_{i,j} \notin c_j \\ 2 & \tilde{l_{i,j}} \in c_j \end{cases}$. It is easy to recognize $m \in \mathbb{Z}_3^{|V|\times k}$ as a $|V| \times k$ matrix with elements taken from $GF(3)$.

It is currently assumed that the formula $i_0^e \cdot i_1^e \cdot \ldots \cdot i_{|v|-1}^e$ represents the *conjunctive normal form* of $F$.

## 6.3 The satisfiability problem as a finite constraint system

Where $S = (I, k, \mathbb{Z}/\mathbb{Z}_3)$ is a finite constraint system the satisfiability problem is defined as finding out if an integer $i \neq 0$ exists such that $I\%i = \emptyset$ .

## 6.4 The generalized satisfiability problem as a finite constraint system

Where $S = (I, k, F)$ is a finite constraint system the generalized satisfiability problem is defined as finding out if an integer $i \neq 0$ exists such that $I\%i = \emptyset$.